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Non-linear resonant instability in boundary layers

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An investigation is made of resonant triads of Tollmien–Schlichting waves in an unstable boundary layer. The triads considered are those comprising a two-dimensional wave and two oblique waves propagating at equal and opposite angles to the flow direction and such that all three waves have the same phase velocity in the downstream direction. For such a resonant triad remarkably powerful wave interations take place, which may cause a continuous and rapid transfer of energy from the primary shear flow to the disturbance. It appears that the oblique waves can grow particularly rapidly and it is suggested that such preferential growth may be responsible for the rapid development of three-dimensionality in unstable boundary layers. The non-linear energy transfer primarily takes place in the vicinity of the critical layer where the downstream propagation velocity of the waves equals the velocity of the primary flow.

The theoretical analysis is initially carried out for a general primary velocity profile; then, in order to demonstrate the essential features of the results, precise interaction equations are derived for a particular profile consisting of a layer of constant shear bounded by a uniform flow. Some exact solutions of the general interaction equations are presented, one of which has the property that the wave amplitudes become indefinitely large at a finite time. The possible relevance of the present theoretical model to the experiments of Klebanoff, Tidstrom & Sargent (1962) is examined.

1. Introduction

The experiments of Klebanoff & Tidstrom (1959) and Klebanoff *et al.* (1962) on the development of three-dimensionality in unstable boundary layers have established the existence of variations, in both the mean and fluctuating velocity components, which are periodic in the spanwise direction and which grow in intensity with distance downstream. In these experiments the spanwise periodicity was in most cases fixed by creating an artificial disturbance with a small spanwise variation in amplitude. However the development of a similar, but less regularly spaced, spanwise structure was also observed for natural transition: accordingly, the growth of such spanwise variations appears to be a consistent and significant feature of the developing instability.

The current state of knowledge of boundary-layer transition is admirably reviewed by Tani (1969). The most successful theoretical attempts to model the observed growth of three-dimensionality are those of Benney & Lin (1960) and

Benney (1961, 1964). They consider the second-order non-linear interaction of a two-dimensional Tollmien–Schlichting wave and a three-dimensional wave with spanwise periodicity (such as is produced by two plane Tollmien–Schlichting waves propagating at equal and opposite angles to the flow direction). They find that this interaction promotes the growth of a secondary system of spanwise-periodic longitudinal vortices which are qualitatively similar to those reported by Klebanoff *et al.* Also, the observed tendency for the periodicity of these vortices to halve as the velocity fluctuations increase in amplitude is reproduced by the Lin–Benney model.

However the Lin–Benney model is incomplete in some important respects. First, it has been pointed out (Stuart 1962*a, b*) that, according to linear theory, the frequency of Tollmien–Schlichting waves depends upon wave-number in such a way that the two- and three-dimensional waves of the Lin–Benney model cannot be coupled in phase, contrary to assumption, but that their frequencies may differ by as much as 15 %. This criticism is considered further in § 2.

Another shortcoming of the Lin–Benney theory is that it furnishes no estimate of a preferred spanwise periodicity; for, by choosing the periodicity of the three-dimensional wave, any desired spacing can be generated. While such a choice may appropriately represent an artificially induced disturbance with built-in spanwise periodicity, the Lin–Benney model cannot determine whether there are three-dimensional disturbances which are particularly susceptible to amplification.

The present paper concerns a non-linear mechanism which may favour the selective growth of such three-dimensional disturbances. This mechanism turns out to be a remarkably strong one, involving resonant interactions among a suitable triad of Tollmien–Schlichting waves. Previous theoretical investigations into the possibility of such resonance have been made by Raetz (1959, 1964) and Stuart (1962*b*); however, in these, the occurrence of resonance was established only for certain triads of waves which are neutrally stable according to linear theory. The restriction of the analysis to such special cases results in the exclusion of the much stronger resonance mechanism revealed in the present paper.

There have been numerous studies of resonant wave interactions in the absence of a primary shear flow, the most general of which is that of Simmons (1969). Such interactions result in an exchange of energy among the participating waves such that the total wave energy is conserved. However, investigations of resonant wave interactions in shear flows by Kelly (1968) and Craik (1968) have established that, in addition to the interchange of energy among the wave components owing to their interaction, a transfer of energy from the primary shear flow to the disturbance (or *vice versa*) may also occur. The energy-transfer mechanism described by Craik (1968) for a resonant triad of gravity waves in a uniform shear flow is shown in the present paper to operate among a triad of Tollmien–Schlichting waves in a boundary layer. As in Craik's previous work, the three-dimensionality of the waves is an essential feature of the mechanism. The (non-linear) exchange of energy between the primary shear flow and the waves is found to take place in the vicinity of the critical layer where the fluid velocity equals the downstream propagation velocity of the waves.

In § 2 the possibility of such resonance among Tollmien–Schlichting waves is examined using data for the Blasius boundary-layer profile kindly supplied by Dr R. Jordinson of Edinburgh University. The non-linear analysis of the resonant interactions is developed in §§ 3 and 4 for a general boundary-layer profile $\bar{u}(z)$, and the illustrative case of the particular (dimensionless) boundary-layer profile $\bar{u} = z (0 \leqslant z \leqslant 1), \bar{u} = 1 (1 \leqslant z \leqslant \infty)$ is examined in § 5. The approximations are discussed in § 6 and some exact solutions of the non-linear equations for the wave amplitudes are derived in § 7. Finally, the possible relevance of the present analysis to the experiments of Klebanoff *et al.* is discussed in § 8.

2. The possibility of resonance

We consider small perturbations of a laminar boundary layer flowing in the x direction and which, locally, may be regarded as a function $\bar{u}(z)$ of the distance z normal to a plane rigid boundary situated at $z = 0$. The perturbations may depend on x, z, the spanwise co-ordinate y and the time t. They are assumed to consist of an assemblage of Tollmien–Schlichting waves with velocity components of the form $\mathrm{Re}\{f(z) \exp i(\alpha x + \beta y - \alpha c t)\}$, where $c = c_r + i c_i$ is the complex wave velocity in the x direction and the wave-number components α, β are real. All quantities are regarded as having been made dimensionless relative to the free-stream velocity V and density ρ of the fluid and the thickness δ of the boundary layer. The appropriate functions $f(z)$ and the complex eigenvalue relationship $c = c(\alpha, \beta, R)$ are obtained by solution of the linearized equations and depend on the Reynolds number $R = V\delta/\nu$, where ν is the kinematic viscosity.

We here examine the results of linearized theory to establish whether three waves with respective x, y, t dependence of the form $\exp i(\alpha' x - \alpha' c' t)$ and $\exp i(\alpha x \pm \beta y - \alpha c t)$ may comprise a resonant triad: that is, we consider a two-dimensional wave and two oblique waves propagating at equal and opposite angles to the x direction. Clearly, resonance occurs at second order only if $\alpha' = 2\alpha$ and $c'_r = c_r$, and choosing α' equal to 2α reduces the problem to a search for values of α and β at given R such that $c_r(\alpha, \beta, R) = c_r(2\alpha, 0, R)$. (Note that for resonance only the real parts of c and c' must be equal since the imaginary parts relate to the growth or decay of the waves.)

The reasons for seeking resonance among waves of this particular form are twofold. First, the work of Craik (1968) has established that resonant triads of this type may interact in a particularly powerful manner owing to the fact that the critical layers for the three waves coincide. (This point is discussed further in § 8.) Second, such a mechanism might lead to the selective amplification of a particular pair of oblique waves which would subsequently impart a preferred spanwise periodicity to the flow by interaction of the Lin–Benney type.

The relevant information for determining criteria for resonance is contained in the results of linear stability theory for two-dimensional disturbances; for the Squire transformation readily yields the result

$$c(\alpha, \beta, R) = c(\gamma, 0, \alpha R/\gamma), \quad \gamma = (\alpha^2 + \beta^2)^{\frac{1}{2}},$$

which expresses the complex wave velocity (in the x direction) of a three-dimensional disturbance in terms of that for a two-dimensional disturbance at lower Reynolds number. Such data for two-dimensional waves in the Blasius boundary layer, obtained by numerical solution of the Orr–Sommerfeld equation, were kindly supplied to the author by Dr R. Jordinson. These may be conveniently displayed as curves of α against R and c_r against R for various constant values of c_i, some of which are shown in figure 1. For these results, δ is taken as the displacement thickness in the definition of R. Similar curves have been published by Kaplan (1964) and Wazzan, Okamura & Smith (1966) (see also Betchov & Criminale 1967, p. 90).

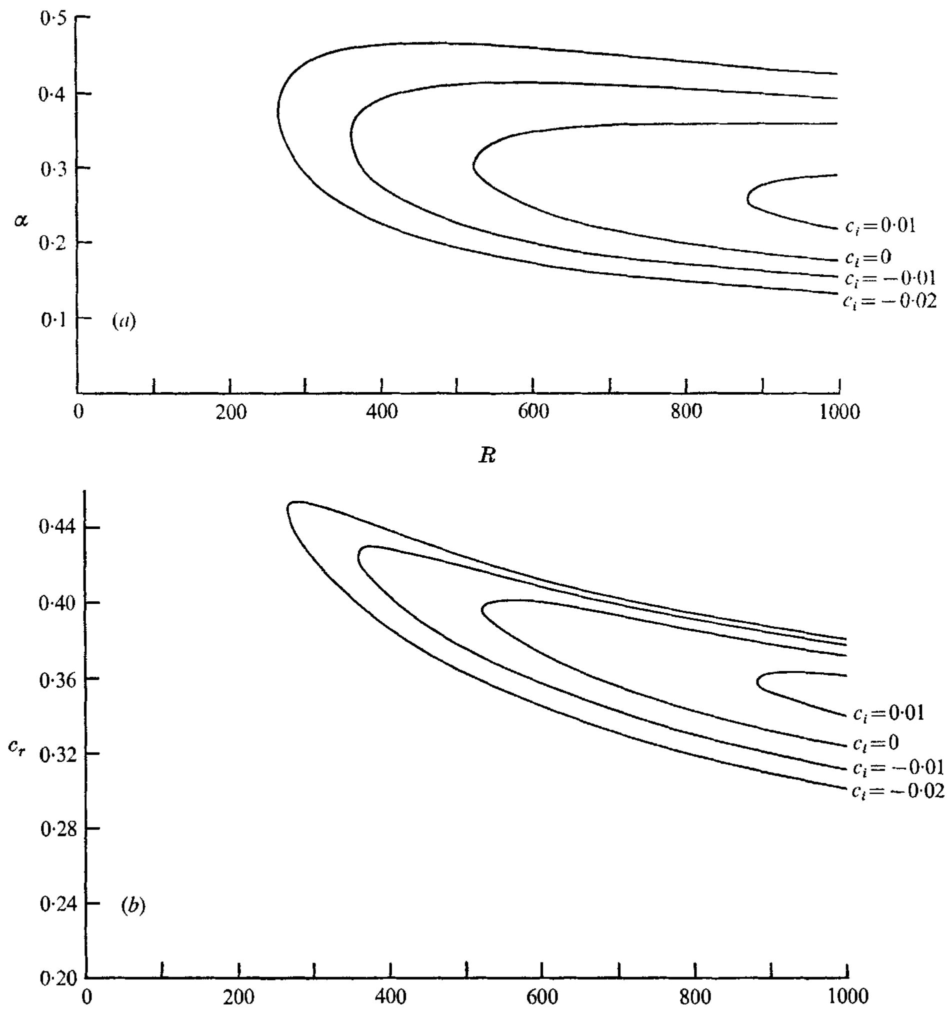

FIGURE. 1 Curves of constant c_i for the Blasius boundary layer.

The presentation of the corresponding results for three-dimensional disturbances follows that of Watson (1960) and Betchov & Criminale (1967, p. 115) and takes the form of curves of constant c_r and c_i at given R on a diagram with the wave-number components α and β as co-ordinates. Such curves, deduced from figure 1 by means of the Squire transformation, are shown in figure 2 for $R = 882$. Similar curves are given by Betchov & Criminale, but only for positive values of c_i. The curves of constant c_r and c_i are of course symmetrical about the axis $\beta = 0$: for convenience those of constant c_r are shown only for $\beta \geqslant 0$ and those of constant c_i for $\beta \leqslant 0$. This set of curves for $R = 882$ is sufficient to demonstrate the existence of resonant triads of Tollmien–Schlichting waves at this Reynolds number. At other Reynolds numbers similar results will hold. The wave with $\alpha = 0{\cdot}254$, $\beta = 0$ has the greatest value of c_i, equal to 0·01; this is close to the most unstable disturbance at $R = 882$. (The most unstable disturbance is actually that for which αc_i is a maximum, but the difference is not large.) Consideration of the curve of constant wave velocity $c_r = 0{\cdot}357$, which passes through the point $\alpha = 0{\cdot}254$, $\beta = 0$, confirms that the waves of the Lin–Benney mechanism cannot be coupled in phase since no three-dimensional wave with $\alpha = 0{\cdot}254$ and β non-zero lies on this curve.

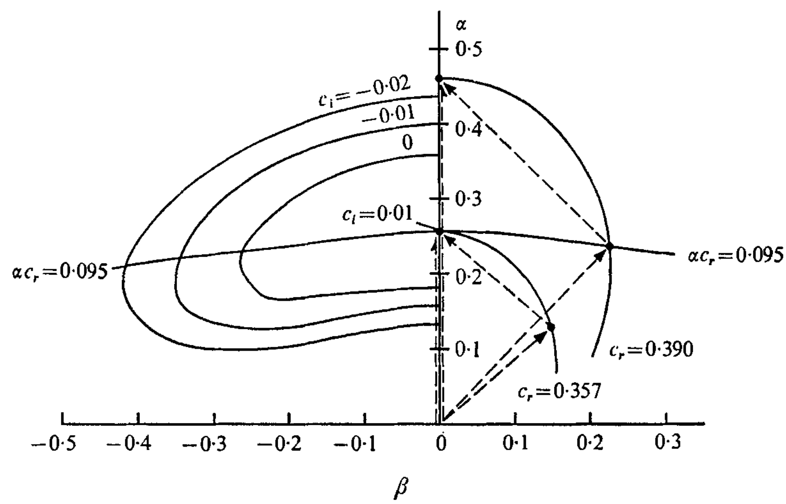

FIGURE 2. Curves of constant c_r and c_i for three-dimensional disturbances with wave-number components (α, β) at $R = 882$. The curves of constant c_i are shown only for $\beta \leqslant 0$ and those of constant c_r for $\beta \geqslant 0$. Both curves are symmetrical about $\beta = 0$. The arrows designate resonant wave triads.

However, this criticism of the Lin–Benney mechanism is not as serious as it might seem. Those disturbances which *are* coupled in phase with the two-dimensional wave with $\alpha = 0{\cdot}254$ are represented on figure 2 by the curve $\alpha c_r = 0{\cdot}095$ and it is seen that this curve 'bends' much less than that for constant c_r. Accordingly, interaction between the two-dimensional wave and any three-dimensional wave represented by a point on this line will produce secondary longitudinal-vortex flows inclined at only a small angle to the x direction. The addition of two such secondary flows, deriving from the interaction of the two-dimensional wave with either of two oblique waves of equal amplitude and propagating at equal and opposite angles to the x direction, gives rise to a longitudinal-vortex flow with a small spatial periodicity in the x direction. (Such a suggestion was put forward by Benney (1964).) It is unlikely that this small periodicity would be detected experimentally within a distance of only a few wavelengths.

It seems therefore that modification of the Lin–Benney model along these lines would ensure that the participating waves are coupled in phase. It is also possible, as was suggested by Klebanoff *et al.*, that synchronization of the waves might be brought about by non-linear effects, but such an explanation is inconsistent with the Lin–Benney quasi-linearization procedure.

The existence of resonance of the desired form is readily verified from figure 2 in the following way. Consider a two-dimensional wave of given wave-number α_0 and phase speed c_r and follow on the α–β diagram the curve on which c_r takes this constant value. If there is a point (α, β) on this line for which $\alpha = \frac{1}{2}\alpha_0$ the waves represented by this point and the corresponding point $(\alpha, -\beta)$ form a resonant triad with the given two-dimensional wave.

Two such triads which may have particular physical significance are shown in figure 2. For the first of these the two-dimensional wave is that with $\alpha_0 = 0{\cdot}254$, which is close to the most unstable disturbance and which is therefore likely to be the largest disturbance present during the initial stages of natural transition. It is readily seen that the points $\alpha = 0{\cdot}127$, $\beta = \pm 0{\cdot}148$ lie at the intersection of the appropriate curve of constant c_r and the line $\alpha = \frac{1}{2}\alpha_0$. Consequently the two oblique waves $\alpha = 0{\cdot}127$, $\beta = \pm 0{\cdot}148$ complete the resonant triad. The wave-number vector diagram for this triad, which is in the form of an isosceles triangle, is indicated by arrows on figure 2.

For the second triad the two-dimensional wave was chosen with frequency exactly twice that of the wave with $\alpha_0 = 0{\cdot}254$, $\beta = 0$ with the result that the resonating oblique waves have the same frequency as the latter wave. The resonant triad of wave-numbers (α, β) in this case is $(0{\cdot}46, 0)$, $(0{\cdot}23, 0{\cdot}23)$, $(0{\cdot}23, -0{\cdot}23)$, and the corresponding wave-number vector diagram is also shown in figure 2. Assuming that resonant interactions among this triad result in preferential amplification of the two oblique waves – which is verified in § 7 – and that the largest two-dimensional wave present is the (linearly most unstable) wave with $\alpha = 0{\cdot}254$, we see that these three waves of the same frequency may then interact in the Lin–Benney manner to impart a definite spanwise periodicity to the flow. Further discussion of these ideas and of their possible significance in the experiments by Klebanoff *et al.* is delayed until § 8 by which stage the nature of the resonant interaction will have been clarified.

It is clear that at any given Reynolds number resonant triads of Tollmien–Schlichting waves may always be found by applying the above procedure.

3. Non-linear analysis

We first consider perturbations of a primary shear flow $\bar{u}(z)$ by a two-dimensional wave defined by the perturbation stream function

$$\psi_3 = \mathrm{Re}\{\phi_3(z)\, A_3(t) \exp[i\alpha(x-ct)]\}, \quad c = c_r + ic_i,$$

with associated velocity perturbations

$$u_3 = \phi_3'(z)\, A_3 \exp[i\alpha(x-ct)], \quad w_3 = -i\alpha\phi_3(z)\, A_3 \exp[i\alpha(x-ct)]$$

in the x and z directions respectively. Henceforth the real parts are taken to represent physical quantities and the dash denotes differentiation with respect to z. Here, A_3 is a measure of the wave amplitude which will be constant according to linear theory but which will be a slowly varying function of time t when non-linear interactions are considered.

Similarly, we consider two plane waves propagating at equal and opposite angles to the x direction with velocity components (u_1, v_1, w_1), (u_2, v_2, w_2) respectively in the x, y and z directions and with x, y, t dependence of the form $\exp[i(\frac{1}{2}\alpha x \pm \beta y - \frac{1}{2}\alpha\tilde{c}t)]$. It is readily shown (cf. Craik 1968) that

$$\gamma u_{1,2} = \tfrac{1}{2}\alpha\hat{u}_{1,2} \mp \beta\hat{v}_{1,2}, \quad \gamma v_{1,2} = \pm\beta\hat{u}_{1,2} + \tfrac{1}{2}\alpha\hat{v}_{1,2}, \quad \gamma = (\tfrac{1}{4}\alpha^2 + \beta^2)^{\frac{1}{2}},$$

where upper and lower signs refer to waves 1 and 2 respectively and $\hat{u}_{1,2}$, $\hat{v}_{1,2}$ are the velocity components in the directions $\hat{x}_{1,2}$, $\hat{y}_{1,2}$ defined by

$$\gamma\hat{x}_{1,2} = \tfrac{1}{2}\alpha x \pm \beta y, \quad \gamma\hat{y}_{1,2} = \mp\beta x + \tfrac{1}{2}\alpha y.$$

Also, from the appropriate equations of continuity it is seen that $\hat{u}_{1,2}$ and $w_{1,2}$ may be expressed in terms of the perturbation stream functions

$$\psi_{1,2} = \operatorname{Re}\{\phi_{1,2}(z)\,A_{1,2}(t)\exp[i(\tfrac{1}{2}\alpha x \pm \beta y - \tfrac{1}{2}\alpha\tilde{c}t)]\}, \quad \tilde{c} = \tilde{c}_r + i\tilde{c}_i$$

as

$$\hat{u}_{1,2} = \phi'_{1,2}(z)\,A_{1,2}\exp[i(\tfrac{1}{2}\alpha x \pm \beta y - \tfrac{1}{2}\alpha\tilde{c}t)],$$

$$w_{1,2} = i\gamma\phi_{1,2}(z)\,A_{1,2}\exp[i(\tfrac{1}{2}\alpha x \pm \beta y - \tfrac{1}{2}\alpha\tilde{c}t)].$$

The linearized vorticity equations for these perturbations are

$$L_3[\phi_3] \equiv i\alpha[(\bar{u}-c)(\phi''_3 - \alpha^2\phi_3) - \bar{u}''\phi_3] - R^{-1}(\phi_3^{\text{iv}} - 2\alpha^2\phi''_3 + \alpha^4\phi_3) = 0, \qquad (3.1\,a,\,b)$$

$$L_{1,2}[\phi_{1,2}] \equiv \tfrac{1}{2}i\alpha[(\bar{u}-\tilde{c})(\phi''_{1,2} - \gamma^2\phi_{1,2}) - \bar{u}''\phi_{1,2}] - R^{-1}(\phi_{1,2}^{\text{iv}} - 2\gamma^2\phi''_{1,2} + \gamma^4\phi_{1,2}) = 0,$$

and the linearized momentum equations in the $\hat{y}_{1,2}$ directions yield

$$[\tfrac{1}{2}i\alpha(\bar{u}-\tilde{c}) - R^{-1}(\partial^2/\partial z^2 - \gamma^2)]\,\hat{v}_{1,2} = \mp i\bar{u}'\beta\phi_{1,2}. \qquad (3.2)$$

The solution of $(3.1\,a,\,b)$ subject to the boundary conditions

$$\phi_i(0) = \phi'_i(0) = 0, \quad \phi_i, \phi'_i \to 0 \ (z \to \infty) \quad (i = 1, 2, 3) \qquad (3.3)$$

is assumed to be known from linear stability theory.

Proceeding to consider the effect of resonant interactions among these waves when $c_r = \tilde{c}_r$, we may write the non-linear vorticity equations corresponding to those above as

$$\left.\begin{aligned} A_3(t)\,L_3[\phi_3] &= -(dA_3/dt)(\phi''_3 - \alpha^2\phi_3) + F_3, \\ A_{1,2}(t)\,L_{1,2}[\phi_{1,2}] &= -(dA_{1,2}/dt)(\phi''_{1,2} - \gamma^2\phi_{1,2}) + F_{1,2}, \end{aligned}\right\} \qquad (3.4\,a,\,b)$$

where

$$F_3\exp[i\alpha(x-ct)] \equiv \left[\frac{\partial}{\partial x}(\mathbf{u}\,.\,\nabla w) - \frac{\partial}{\partial z}(\mathbf{u}\,.\,\nabla u)\right]_3,$$

$$F_{1,2}\exp[i(\tfrac{1}{2}\alpha x \pm \beta y - \tfrac{1}{2}\alpha\tilde{c}t)] \equiv \left[\frac{\partial}{\partial\hat{x}_{1,2}}(\mathbf{u}\,.\,\nabla w) - \gamma^{-1}\frac{\partial}{\partial z}\{\mathbf{u}\,.\,\nabla(\tfrac{1}{2}\alpha u \pm \beta v)\}\right]_{1,2}$$

represent the second-order contributions of appropriate periodicity deriving from the non-linear inertia terms of the equations of motion.

F_1, F_2, and F_3 may be evaluated to the required order using the estimates of linear theory for u, v and w. Also, since the rates of change in amplitude $|dA_i/dt|$ are assumed small compared with $|\alpha c A_i|$, the terms $(\phi_3'' - \alpha^2\phi_3)$ and $(\phi_{1,2}'' - \gamma^2\phi_{1,2})$ on the right-hand sides of (3.4a, b) may be evaluated from linear theory to the same order. After some reduction the F_i are found to be

$$\begin{aligned}F_3 = {}& \tfrac{1}{4}i\alpha A_1 A_2 \exp\left[\alpha(\tilde{c}_i - c_i)t\right]\left[\phi_1(\phi_2'' - \gamma^2\phi_2) + \phi_2(\phi_1'' - \gamma^2\phi_1)\right]' \\ & - (\alpha^2 - 2\gamma^2)\gamma^{-2}\left[\phi_1'(\phi_2'' - \gamma^2\phi_2) + \phi_2'(\phi_1'' - \gamma^2\phi_1)\right] \\ & - 2\alpha\beta\gamma^{-2}\left[\hat{v}_2(\phi_1'' - \gamma^2\phi_1) - \hat{v}_1(\phi_2'' - \gamma^2\phi_2) + (\phi_1'\hat{v}_2' - \phi_2'\hat{v}_1')\right] + 4\beta^2\gamma^{-2}(\hat{v}_1\hat{v}_2)' \\ & + 2\beta\alpha^{-1}\left[\phi_1\hat{v}_2 - \phi_2\hat{v}_1\right]''\}, \end{aligned} \qquad (3.5a)$$

$$\begin{aligned}F_2 = {}& \tfrac{1}{4}i\alpha A_3 A_1^* \exp\left[\alpha c_i t\right]\{(\alpha^2 - 2\gamma^2)\gamma^{-2}\phi_3(\phi_1^{*\prime\prime} - \gamma^2\phi_1^*)' \\ & + (\alpha^2 - 3\gamma^2)\gamma^{-2}\phi_3'(\phi_1^{*\prime\prime} - \gamma^2\phi_1^*) - 2\phi_1^{*\prime}(\phi_3'' - \alpha^2\phi_3) - \phi_1^*(\phi_3'' - \alpha^2\phi_3)' \\ & - 2\alpha\beta\gamma^{-2}(\phi_3\hat{v}_1^{*\prime\prime} + \phi_3'\hat{v}_1^{*\prime} + \gamma^2\phi_3\hat{v}_1^*)\}, \end{aligned} \qquad (3.5b)$$

$$\begin{aligned}F_1 = {}& \tfrac{1}{4}i\alpha A_3 A_2^* \exp\left[\alpha c_i t\right]\{(\alpha^2 - 2\gamma^2)\gamma^{-2}\phi_3(\phi_2^{*\prime\prime} - \gamma^2\phi_1^*)' \\ & + (\alpha^2 - 3\gamma^2)\gamma^{-2}\phi_3'(\phi_2^{*\prime\prime} - \gamma^2\phi_2^*) - 2\phi_2^{*\prime}(\phi_3'' - \alpha^2\phi_3) - \phi_2^*(\phi_3'' - \alpha^2\phi_3)' \\ & + 2\alpha\beta\gamma^{-2}(\phi_3\hat{v}_2^{*\prime\prime} + \phi_3'\hat{v}_2^{*\prime} + \gamma^2\phi_3\hat{v}_2^*)\}. \end{aligned} \qquad (3.5c)$$

In these expressions the asterisk denotes complex conjugates which arise owing to the multiplication rule $\mathrm{Re}\{A\}\,\mathrm{Re}\{B\} = \frac{1}{2}\mathrm{Re}\{AB + AB^*\}$.

The homogeneous boundary conditions (3.3) are also applicable to the second-order equations (3.4a, b). Accordingly, in order that non-trivial solutions of these equations may exist, it is necessary that their right-hand sides – evaluated by linear theory – be orthogonal to the solutions of the associated adjoint homogeneous equations (see Ince 1956, §9.34) which satisfy the same homogeneous boundary conditions. These adjoint equations are simply the adjoint Orr–Sommerfeld equations

$$[(\bar{u} - c)\psi_3]'' - \alpha^2(\bar{u} - c)\psi_3 - \bar{u}''\psi_3 - (i\alpha R)^{-1}(\psi_3^{\mathrm{iv}} - 2\alpha^2\psi_3'' + \alpha^4\psi_3) = 0, \quad (3.6a)$$

$$[(\bar{u} - \tilde{c})\psi_{1,2}]'' - \gamma^2(\bar{u} - \tilde{c})\psi_{1,2} - \bar{u}''\psi_{1,2} - (\tfrac{1}{2}i\alpha R)^{-1}(\psi_{1,2}^{\mathrm{iv}} - 2\gamma^2\psi_{1,2}'' + \gamma^4\psi_{1,2}) = 0, \quad (3.6b)$$

which are discussed in some detail by Reid (1965). Assuming that we know the solutions $\psi_{1,2}$ and ψ_3 of these equations which satisfy the appropriate homogeneous boundary conditions, the orthogonality conditions necessary for the existence of solutions of the second-order equations become

$$\left.\begin{aligned}\frac{dA_3}{dt}\int_0^\infty \psi_3(\phi_3'' - \alpha^2\phi_3)\,dz &= \int_0^\infty F_3\psi_3\,dz, \\ \frac{dA_{1,2}}{dt}\int_0^\infty \psi_{1,2}(\phi_{1,2}'' - \gamma^2\phi_{1,2})\,dz &= \int_0^\infty F_{1,2}\psi_{1,2}\,dz,\end{aligned}\right\} \qquad (3.7a, b)$$

where all the integrands may be evaluated from linear theory using the solutions of equations (3.1a, b), (3.2) and (3.6a, b).

4. Asymptotic theory for large αR

The integrals occurring in (3.7*a*, *b*) may be evaluated by computer if precise results are required for a given velocity profile. However in the present paper approximate estimates of these integrals are derived using the results of asymptotic stability theory at large Reynolds numbers. In view of the agreement between such asymptotic linear stability theory and experiment (see Shen 1954; Ross *et al.* 1970) even at Reynolds numbers as low as 500 for the Blasius boundary layer the present use of this approximation seems reasonable. Such an approach has the additional benefit of demonstrating the powerful contribution to the interaction integrals of (3.7*b*) which derives from the vicinity of the critical layer. Subsequent consideration of these results in § 8 will show that the particular class of resonant triads examined comprises those most susceptible to rapid amplification.

At sufficiently large Reynolds numbers, the linear solutions

$$\phi_i, \psi_i, \hat{v}_{1,2} (i = 1, 2, 3)$$

may be adequately represented by inviscid estimates except close to the boundary $z = 0$ and near the 'critical layer' where the velocity of the primary flow equals the phase velocity c_r of the waves. In the latter case the inviscid solutions normally become singular at the point z_c in the complex z plane where $\bar{u}(z_c) = c$ and, since c_i is small, this singularity lies close to the real axis. It is also known from linear theory that the inviscid solutions for ϕ_i, ψ_i and $\hat{v}_{1,2}$ are valid asymptotic approximations in the region of the complex z plane for which

$$-\tfrac{7}{6}\pi < \arg(z-z_c) < \tfrac{1}{6}\pi,$$

excluding a small circle of radius $O\{(\alpha R)^{-\frac{1}{3}}\}$ with centre at z_c.

Clearly the inviscid estimates of the integrand occurring in (3.7*a*, *b*) are also usually singular at z_c but, by using the last mentioned property of the inviscid solutions, the integrals of (3.7*a*) and those on the left-hand side of (3.7*b*) may be evaluated by deforming the path of integration to pass beneath the singularity at z_c. In particular we note that, since the inviscid estimates for ϕ_i and ψ_i are such that $\psi_3 = (\bar{u}-c)^{-1}\phi_3$ and $\psi_{1,2} = (\bar{u}-\tilde{c})^{-1}\phi_{1,2}$, the integrals on the left-hand sides of (3.7*a*, *b*) may be expressed more simply as

$$\left.\begin{aligned} \int_0^\infty \psi_3(\phi_3''-\alpha^2\phi_3)\,dz &= \oint_0^\infty \bar{u}''\psi_3^2\,dz, \\ \int_0^\infty \psi_{1,2}(\phi_{1,2}''-\gamma^2\phi_{1,2})\,dz &= \oint_0^\infty \bar{u}''\psi_{1,2}^2\,dz, \end{aligned}\right\} \qquad (4.1\,a, b)$$

where inviscid estimates for ψ_i are used and the path of integration passes beneath the singularity at z_c.

However, as in the case examined by Craik (1968) this device cannot be used to evaluate the integrals on the right-hand side of (3.7*b*). These integrands involve not only ϕ_i, ψ_i and $\hat{v}_{1,2}$ but also their complex conjugates and it is readily confirmed that the inviscid estimates of these complex-conjugate quantities are valid in the region of the complex z plane denoted by

$$-\tfrac{1}{6}\pi < \arg(z-z_c) < \tfrac{7}{6}\pi.$$

Accordingly it is not possible to deform the path of integration in such a way that the inviscid estimates of the integrand remain valid approximations. Instead, the path of integration must pass through the singularity at z_c and viscous theory must be employed to evaluate the integrands in the vicinity of the critical layer.

The viscous solutions for ϕ_t in the vicinity of the critical layer are well known to be expressible in terms of the Airy function with

$$Z \equiv i(\tfrac{1}{2}\alpha R\overline{u}_c')^{\frac{1}{3}}(z-z_c) \tag{4.2}$$

as the independent variable (see for example Reid 1965), but the viscous solutions for $\psi_{1,2}$ are less familiar and are now derived in more detail.

Near z_c it is easily shown that, to highest order in $(\alpha R)^{\frac{1}{3}}$, the equation for $\psi_{1,2}$ becomes

$$\frac{d^2}{dZ^2}\left(\frac{d^2}{dZ^2}+Z\right)\psi_{1,2} = 0,$$

together with boundary conditions which ensure that the viscous solution matches the inviscid solution away from the critical layer. On integrating twice we obtain the equation

$$\left(\frac{d^2}{dZ^2}+Z\right)\Xi = -1, \quad \Xi \equiv B^{-1}(\psi_{1,2}-A),$$

where A and B are constants of integration. Also, a particular solution of this equation for which $\Xi \to Z^{-1}$ as $Z \to \pm i\infty$ is the Lommel function $L(Z)$ (see Benney 1961, 1964; Craik 1968); and, in virtue of the matching conditions, the Airy function solutions of the homogeneous equation cannot contribute to $\psi_{1,2}$. Accordingly

$$\psi_{1,2} = A + BL(Z).$$

Near z_c the inviscid solutions for $\phi_{1,2}$ have the form (see, for example, Reid 1965)

$$\phi_{1,2} \sim C_{1,2}\{1+(\overline{u}_c''/\overline{u}_c')(z-z_c)\log(z-z_c)\}+O(z-z_c), \quad \text{as} \quad z\to z_c.$$

Therefore, since $\psi_{1,2} = (\overline{u}-\overline{c})^{-1}\phi_{1,2}$ according to inviscid theory, the matching conditions for the above viscous solutions are

$$\psi_{1,2}/C_{1,2} \sim i(\tfrac{1}{2}\alpha R)^{\frac{1}{3}}\overline{u}_c'^{-\frac{2}{3}}Z^{-1}+(\overline{u}_c''/\overline{u}_c'^2)\log[-i(\tfrac{1}{2}\alpha R\overline{u}_c')^{-\frac{1}{3}}Z], \quad \text{as} \quad Z\to \pm i\infty$$

and the corresponding solutions are

$$\psi_{1,2}/C_{1,2} = i(\tfrac{1}{2}\alpha R)^{\frac{1}{3}}\overline{u}_c'^{-\frac{2}{3}}L(Z)+O\{(\overline{u}_c''/\overline{u}_c'^2)\log(\alpha R\overline{u}_c')^{\frac{1}{3}}\}, \tag{4.3}$$

where $C_{1,2}$ denote the values of $\phi_{1,2}(z_c)$ given by inviscid theory.

For $\hat{v}_{1,2}$ the inviscid approximation to (3.2) yields

$$\hat{v}_{1,2} = (\mp 2\beta\overline{u}'/\alpha(\overline{u}-\tilde{c}))\,\phi_{1,2}, \tag{4.4}$$

which is usually singular at z_c. The corresponding viscous solutions for $\hat{v}_{1,2}$ in the vicinity of z_c must satisfy – to highest order in $(\alpha R)^{\frac{1}{3}}$ – the equation

$$(d^2/dZ^2+Z)\,\hat{v}_{1,2} = \mp(2i\beta/\alpha)\,(\tfrac{1}{2}\alpha R\overline{u}_c')^{\frac{1}{3}}\phi_{1,2}.$$

The appropriate solution is found to be (cf. Benney 1964, Craik 1968)

$$\hat{v}_{1,2}/C_{1,2} = \mp(2i\beta/\alpha)\,(\tfrac{1}{2}\alpha R\overline{u}_c')^{\frac{1}{3}}L(Z)+O\{(\overline{u}_c''/\overline{u}_c')\log(\alpha R\overline{u}_c')^{\frac{1}{3}}\}. \tag{4.5}$$

It is easy to confirm that similar results hold for the complex-conjugate quantities $\psi^*_{1,2}$, $\hat{v}^*_{1,2}$ but that $L(Z)$ is replaced by $L(-Z)$ when Z is a purely imaginary quantity. For example, the result corresponding to (4.5) is (cf. Craik 1968)

$$\hat{v}^*_{1,2}/C^*_{1,2} = \pm(2i\beta/\alpha)(\tfrac{1}{2}\alpha R\bar{u}'_c)^{\frac{1}{3}}L(-Z) + O\{(\bar{u}''_c/\bar{u}'_c)\log(\alpha R\bar{u}'_c)^{\frac{1}{3}}\}, \quad (Z \text{ imaginary}). \tag{4.6}$$

Use of the above results will allow evaluation of the integrals in (3.7a, b) for particular boundary-layer profiles. Owing to the singular behaviour of the inviscid estimate of the integrand on the right-hand side of (3.7b) and to the fact that the contour of integration cannot be deformed in a suitable manner to avoid this singularity, it turns out that this integral is dominated by the contribution from the vicinity of the critical layer. However for the other integrals contributions from the whole range of integration may remain significant.

Examination of expressions (3.5b, c) for $F_{1,2}$ reveals that near z_c the terms in $\phi_3\hat{v}^{*\prime\prime}_{2,1}$ are dominant. For the above results show that such terms are $O\{C_3C^*_{2,1}\alpha R\bar{u}'_c\}$ while the largest of the remainder are $O\{C_3C^*_{2,1}(\bar{u}''_c/\bar{u}'_c)(\alpha R\bar{u}'_c)^{\frac{2}{3}}\log(\alpha R\bar{u}'_c)^{\frac{1}{3}}\}$, or perhaps $O\{C_3C^*_{2,1}(\alpha R\bar{u}'_c)^{\frac{2}{3}}\}$ if $\bar{u}''_c$ is small. Consequently, to highest order in $(\alpha R)^{\frac{1}{3}}$, we find that

$$\int_0^\infty F_{1,2}\psi_{1,2}dz = A_3A^*_{2,1}\exp(\alpha c_i t)C_3C^*_{2,1}C_{1,2}[i\pi R(\alpha^2\beta^2/2\gamma^2) + O\{\alpha\bar{u}''_c(\alpha R\bar{u}'_c)^{\frac{2}{3}}\log(\alpha R\bar{u}'_c)^{\frac{1}{3}}\}], \tag{4.7}$$

on using the result (see Craik 1968, §7)

$$\int_{-i\infty}^{i\infty} L(Z)\frac{d^2L(-Z)}{dZ^2}dZ = \pi i.$$

We note that C_j denotes the value of $\phi_j(z_c)$ given by inviscid theory.

A remarkable feature of this result is that these interaction integrals are directly proportional to the Reynolds number R and are $O(A_3A^*_{1,2}R)$ compared with an interaction integral for the two-dimensional wave which is typically $O(A_1A_2)$. This fact is best illustrated with reference to a particular velocity profile.

5. A particular velocity profile

Since the curvature of the velocity profile does not crucially affect the highest order term in (4.7) a convenient profile is

$$\bar{u} = z\ (0 \leqslant z \leqslant 1), \quad \bar{u} = 1 \quad (1 < z \leqslant \infty). \tag{5.1}$$

Though physically unrealistic and yielding results for linear stability totally unlike those for the Blasius boundary layer it satisfactorily exhibits the non-linear features under discussion. From linear inviscid theory (Tietjens 1925),

$$\phi_3 = (\bar{u}-c)\psi_3 = \begin{cases} e^{-\alpha}\sinh\alpha z/\sinh\alpha & (0 \leqslant z \leqslant 1), \\ e^{-\alpha z} & (1 < z \leqslant \infty), \end{cases}$$

$$\phi_{1,2} = (\bar{u}-\tilde{c})\,\psi_{1,2} = \mp\frac{\alpha(\bar{u}-\tilde{c})}{2\beta\bar{u}'}\hat{v}_{1,2} = \begin{cases} \dfrac{e^{-\gamma}\sinh\gamma z}{\sinh\gamma} & (0 \leqslant z \leqslant 1), \\ e^{-\gamma} & (1 < z \leqslant \infty), \end{cases}$$

$$c = c_r = 1-\tfrac{1}{2}\alpha^{-1}(1-e^{-2\alpha}), \quad \tilde{c} = \tilde{c}_r = 1-\tfrac{1}{2}\gamma^{-1}(1-e^{-2\gamma}).$$

The condition for resonance is satisfied when $\gamma = \alpha$: that is, when the oblique waves propagate at angles of $\pm\frac{1}{3}\pi$ to the x direction. Indeed it is readily shown that whenever the wave velocities c and $\tilde{c}$ are determined by inviscid theory, a resonant triad of this type, comprising three wave-number vectors forming an equilateral triangle, must always exist. Henceforth we take $\gamma = \alpha$ and $\beta = \frac{1}{2}\sqrt{3}\alpha$.

Results (4.1 a, b) here give

$$\int_0^\infty \psi_i(\phi_i''-\alpha^2\phi_i)\,dz = -\alpha^2\,\mathrm{cosech}^2\,\alpha \quad (i = 1,2,3)$$

and, on considering the viscous solutions near $z = 0$, it is easily shown that the viscous correction to this result is $O\{(\alpha R)^{-\frac{1}{3}}\}$.

The highest order viscous correction to the complex phase velocity c is obtained by adding the term $(1+i)\,(2\alpha Rc_r)^{-\frac{1}{2}}e^{-2\alpha}\cosh\alpha$ (the factor of $\cosh\alpha$ was erroneously omitted by Tietjens). Therefore

$$c_i = (2\alpha Rc_r)^{-\frac{1}{2}}e^{-2\alpha}\cosh\alpha, \tag{5.2}$$

and result (4.7) yields

$$\int_0^\infty F_{1,2}\psi_{1,2}\,dz = A_3A_{2,1}^*\exp[\alpha c_i t]\left(\frac{e^{-\alpha}\sinh\alpha c_r}{\sinh\alpha}\right)^3\tfrac{3}{8}i\pi\alpha^2R$$

to highest order in $(\alpha R)^{\frac{1}{3}}$, where c_r and c_i are known.

The evaluation of the interaction integral in (3.7 a) is straightforward but tedious. After reduction it is found to be

$$\int_0^\infty F_3\psi_3\,dz = A_1A_2\exp[\alpha(\tilde{c}_i-c_i)\,t]\,\alpha^3\,\mathrm{cosech}^2\,\alpha(\Lambda_r+i\Lambda_i),$$

where

$$\begin{aligned}\Lambda_r = -\tfrac{1}{4}\alpha^2\,\mathrm{cosech}\,\alpha\{&\tfrac{5}{8}-\tfrac{3}{8}e^{-2\alpha}+\tfrac{3}{4}e^{\alpha}(1-\alpha e^{-\alpha}\,\mathrm{cosech}\,\alpha)^{-2} \\ &-\alpha^{-1}(1-c)^{-1}e^{-3\alpha}(\tfrac{9}{2}\sinh\alpha-\tfrac{3}{2}\sinh 3\alpha-\tfrac{3}{4}e^{\alpha}\sinh^2\alpha)+\tfrac{3}{4}\alpha^{-1}c^{-1}e^{-\alpha}\sinh^2\alpha \\ &-\tfrac{9}{4}[e^{3\alpha c}(Ei\langle 3\alpha(1-c)\rangle+E_1\langle 3\alpha c\rangle)-e^{-3\alpha c}(Ei\langle 3\alpha c\rangle+E_1\langle 3\alpha(1-c)\rangle) \\ &-e^{\alpha c}(Ei\langle\alpha(1-c)\rangle+E_1\langle\alpha c\rangle)+e^{-\alpha c}(1+\tfrac{1}{3}e^{2\alpha}\sinh^2\alpha)\,(Ei\langle\alpha c\rangle+E_1\langle\alpha(1-c)\rangle)]\},\end{aligned}$$

$$\Lambda_i = \tfrac{9}{8}\pi\alpha^2\,\mathrm{cosech}\,\alpha\,e^{-3\alpha}\{\cosh(3\alpha c)-\cosh(\alpha c)-\tfrac{1}{6}e^{\alpha(2-c)}\sinh^2\alpha\},$$

$$E_1\langle\eta\rangle \equiv \int_\eta^\infty\frac{e^{-t}dt}{t}, \quad Ei\langle\eta\rangle \equiv \mathrm{P}\int_{-\infty}^{\eta}\frac{e^t\,dt}{t},$$

$$\tilde{c}_i-c_i = (2\alpha Rc_r)^{-\frac{1}{2}}e^{-2\alpha}\cosh\alpha(2^{\frac{1}{2}}-1).$$

Here, P denotes the principal part and $E_1\langle\eta\rangle$, $Ei\langle\eta\rangle$ are tabulated functions (see Abramowitz & Stegun 1964). In this case the interaction integral is complex owing to the contribution of logarithmic terms on integrating under the singularity at z_c. There is no explicit dependence on R, except in the expression for $\tilde{c}_i - c_i$, and so, as mentioned above, the integral is of magnitude $O(A_1 A_2)$ compared with integrals of order $O(A_3 A^*_{2,1} R)$ for the oblique waves 1 and 2.

Equations (3.7 a, b) are therefore

$$\alpha^{-1} dA_3/dt = (\Lambda_r + i\Lambda_i) A_1 A_2 \exp[\alpha(\tilde{c}_i - c_i) t], \quad \alpha^{-1}(dA_{1,2}/dt)\, i\Theta A_3 A^*_{2,1} \exp(\alpha c_i t),$$
$$\Theta \equiv \tfrac{3}{8}\pi R\alpha^{-1} e^{-3\alpha} \operatorname{cosech} \alpha \sinh^3 \alpha c_r,$$

where Λ_r, Λ_i and c_r are given above. It is convenient to re-express these results in terms of the dimensionless complex wave amplitudes a_i ($i = 1, 2, 3$) defined as

$$a_{1,2}(t) = 2c_r^{-1} A_{1,2}(t) \exp[\tfrac{1}{2}\alpha \tilde{c}_i t], \quad a_3(t) = c_r^{-1} A_3(t) \exp[\alpha c_i t].$$

Then the above equations become

$$\left.\begin{aligned} da_3/d(\alpha t) &= c_i a_3 + \tfrac{1}{4} c_r^2 \Lambda a_1 a_2, \quad \Lambda \equiv \Lambda_r + i\Lambda_i, \\ da_{1,2}/d(\alpha t) &= \tfrac{1}{2}\tilde{c}_i a_{1,2} + \tfrac{1}{2} c_r^2 i\Theta a_3 a^*_{2,1}. \end{aligned}\right\} \tag{5.3 a, b}$$

The most surprising feature of these results, as for those of Craik (1968), is the size of the interaction coefficient $|\Theta|$ for the oblique waves, which is $O(R)$, compared with that of $|\Lambda|$ for the two-dimensional wave, which is $O(1)$. This means that when the three a_i are of comparable size the interaction terms affecting the oblique waves are an order of magnitude larger than that which influences the two-dimensional wave. The large magnitude of the former terms has been shown to derive wholly from the vicinity of the critical layer. Evidently, a remarkably strong non-linear energy-transfer mechanism operates in this region. The operation of this mechanism depends crucially on the three-dimensional character of the waves since the dominant contributions to the interaction integrals here derive from terms involving the 'cross-velocity' components $\hat{v}^*_{1,2}$ which are parallel to the respective wave crests.

For more general velocity profiles which exhibit non-zero curvative at the critical layer these remarks will remain valid; for, as indicated in §4, terms proportional to $\bar{u}''_c$ in the interaction integrals for waves 1 and 2 are small compared with that retained above, provided the Reynolds number is large enough. However for velocity profiles such that the phase velocity c_r tends to zero as αR tends to infinity – but such that the asymptotic viscous analysis of §4 remains valid – there may be an additional implicit dependence on R through the quantities C_j of (4.7). At the present time the importance of this contribution to the interaction coefficients has not been evaluated, but it seems likely that the coefficients for the oblique waves will remain large.

6. The approximations

At this stage it is appropriate to make a more careful examination of the approximations underlying the analysis so far. The basic assumption that inviscid theory yields a valid first approximation over most of the flow field, and that asymptotic methods may be employed to develop viscous solutions near $z = 0$ and z_c, is valid provided $(\alpha R)^{\frac{1}{3}} \gg 1$. Also, the quasi-linearization procedure whereby non-linear terms are evaluated by means of linear theory is justified provided the respective wave amplitudes are sufficiently small. In most non-linear analyses, the condition that the squares of the wave slopes should remain small – that is $|\alpha a_i|^2 \ll 1$ – is normally sufficient to ensure that the time scale associated with the non-linear modulation of the waves is large compared with the wave periods. However, in view of the remarkably strong non-linear interactions which affect the oblique waves, here we require $\Theta|a_3|$ to be small compared with unity in order that this property is satisfied. The equivalent condition for the two-dimensional wave is $|\Lambda a_1 a_2|/|a_3| \ll 1$. These conditions, which may be rewritten as

$$(\alpha R)^{\frac{1}{3}} \gg 1, \quad |a_3 R| \ll 1, \quad |a_1 a_2|/|a_3| \ll 1, \qquad (6.1\,a, b, c)$$

define the range of validity of the present analysis. The second of these conditions is particularly severe for it requires the (dimensionless) amplitude of the two-dimensional wave to be $o(R^{-1})$, and R is typically large. This restriction is a direct result of the strength of the non-linear mechanism since, with larger two-dimensional waves, the predicted time scale for the non-linear evolution of the oblique waves is of comparable magnitude with the wave period.

It is worth noting that the foregoing analysis for the particular profile (5.1) remains valid when the linear and non-linear terms on the right-hand sides of (5.3a, b) are of comparable magnitude. This is so because $|c_i|$ and $|\tilde{c}_i|$ are $O\{(\alpha R)^{-\frac{1}{3}}\}$, in virtue of result (5.2), and the estimates of linear theory used to evaluate the non-linear interaction terms are expressed in terms of the inviscid phase velocity $c = c_r$ to the required degree of accuracy. However in situations where $|c_i/c_r|$ is not small the eigenfunctions ϕ, ψ and $\hat{v}_{1,2}$ will depend significantly on c_i as well as on c_r, and it may be necessary to insist that the non-linear interaction terms remain small compared with the linear growth terms in the equations equivalent to (5.3a, b).

The present theory neglects all third-order wave interactions such as those examined by Stuart (1962a, b) and others. For the non-resonant case third-order interactions provide the largest non-linear terms in the interaction equations; but here they are likely to be small compared with the second-order terms. For example, if the third-order interaction parameter is $O(1)$ it is sufficient to have

$$|a_1 a_2 a_3| \gg |\Lambda|^{-1}|a_3|^4, \quad |\Theta|^{-1}|a_{1,2}|^4.$$

Recalling that $|\Theta|$ is $O(R)$ and $|\Lambda|$ is $O(1)$, it is readily shown that these conditions are satisfied in virtually all cases where the inequalities (6.1a, b, c) hold. An exception is that where $|a_1|$ and $|a_2|$ are very small compared with $|a_3|$. For this case the third-order terms in $|a_3|^3$ should be retained in the equation for da_3/dt – but not in those for $da_{1,2}/dt$ – until the growth of $|a_1|$ and $|a_2|$ brings about the dominance of the second-order term. Indeed, it would seem that when resonance can occur the validity of non-resonant third-order theories may be restricted to just this case. For, given a two-dimensional wave two initially small oblique waves completing the resonant triad may grow very rapidly and eventually dominate the third-order effects.

7. Some particular solutions

Before going on to discuss the possible significance of the above results within the context of boundary-layer transition, it is worthwhile to consider in more general terms the behaviour of some solutions of interaction equations of the type (5.3*a*, *b*). We consider the equations

$$\left.\begin{aligned} da_3/dt &= \sigma a_3 + \lambda a_1 a_2, \\ da_1/dt &= \tilde{\sigma} a_1 + i\mu a_3 a_2^*, \quad da_2/dt = \tilde{\sigma} a_2 + i\mu a_3 a_1^*, \end{aligned}\right\} \tag{7.1}$$

where σ, $\tilde{\sigma}$ and μ are real constants and λ is complex. Equations (5.3*a*, *b*) are of this form with

$$\sigma = \alpha c_i, \quad \tilde{\sigma} = \tfrac{1}{2}\alpha\tilde{c}_i, \quad \lambda = \tfrac{1}{4}\alpha c_r^2 \Lambda, \quad \mu = \tfrac{1}{2}\alpha c_r^2 \Theta.$$

First, we note that when $\mu \gg |\lambda|$, as it is above, and σ, $\tilde{\sigma}$ are sufficiently small, these equations may be approximated by

$$da_3/dt = 0, \quad da_1/dt = i\mu a_3 a_2^*, \quad da_2/dt = i\mu a_3 a_1^*.$$

These have the exact solution

$$a_3 = a_{30}, \quad a_1 = a_{10}\cosh(\mu|a_{30}|t) + ia_{20}^*(a_{30}/|a_{30}|)\sinh(\mu|a_{30}|t),$$
$$a_2 = a_{20}\cosh(\mu|a_{30}|t) + ia_{10}^*(a_{30}/|a_{30}|)\sinh(\mu|a_{30}|t),$$

where a_{10}, a_{20}, a_{30} are constants. For this solution the two-dimensional wave has constant amplitude and (excluding the special case $|a_{10}| = |a_{20}|$, $a_{30}/|a_{30}| = ia_{10}a_{20}/|a_{20}|^2$) both oblique waves ultimately grow like $\exp(\mu|a_{30}|t)$. It is seen from (6.1*a*, *b*, *c*) that if

$$(\alpha R)^{\frac{1}{3}} \gg 1, \quad (\alpha R)^{-\frac{1}{3}} \ll |a_{30}|R \ll 1, \quad (|a_{10}|\,|a_{20}|/|a_{30}|)\exp(R|a_{30}|t) \ll 1,$$

this approximation is appropriate for (5.3*a*, *b*) (recall that $\tilde{c}_i$ is $O\{(\alpha R)^{-\frac{1}{3}}\}$). Clearly therefore, the non-linear resonant interaction provides a mechanism for selective amplification of the pair of oblique waves, even in situations where such waves may be damped according to linear theory.

Returning to the full equations (7.1), we see that the change of variables

$$B_3 = i\mu a_3, \quad B_{1,2} = \mu^{\frac{1}{2}}|\lambda|^{\frac{1}{2}}a_{1,2}, \quad e^{i\theta} = i\lambda/|\lambda|,$$

simplifies these to

$$\left.\begin{aligned} \dot{B}_3 &= \sigma B_3 + e^{i\theta}B_1 B_2, \\ \dot{B}_1 &= \tilde{\sigma}B_1 + B_3 B_2^*, \quad \dot{B}_2 = \tilde{\sigma}B_2 + B_3 B_1^*, \end{aligned}\right\} \tag{7.2}$$

where the dot notation has been introduced to denote differentiation with respect to t. Equations of this form for the case $\sigma = \tilde{\sigma} = 0$ have been discussed by Craik (1968, §9) who found that, except when $\theta = \pi$, there always exist solutions for which the wave energy grows without bound. The reader is referred to Craik's paper for details.

Here we consider some particular solutions of (7.2) with the aim of acquiring a further understanding of the role of non-linearity. We may write

$$B_3 = e^{\sigma t} b_3 e^{i\chi_3}, \quad B_{1,2} = e^{\tilde{\sigma} t} b_{1,2} e^{i\chi_{1,2}},$$

where b_1, b_2, b_3 are real and non-negative and χ_1, χ_2, χ_3, are the phases of the complex amplitudes (noting that the linearized equations have solutions with the b_i and χ_i constant). Then the real and imaginary parts of the above three complex equations yield the six results

$$\left.\begin{aligned} \dot{b}_3 &= e^{(2\tilde{\sigma}-\sigma)t} b_1 b_2 \cos(\theta-\chi), \quad \dot{b}_1 = e^{\sigma t} b_3 b_2 \cos\chi, \quad \dot{b}_2 = e^{\sigma t} b_3 b_1 \cos\chi, \\ b_3\dot{\chi}_3 &= e^{(2\tilde{\sigma}-\sigma)t} b_1 b_2 \sin(\theta-\chi), \quad b_1\dot{\chi}_1 = e^{\sigma t} b_3 b_2 \sin\chi, \quad b_2\dot{\chi}_2 = e^{\sigma t} b_3 b_1 \sin\chi, \end{aligned}\right\} \tag{7.3}$$

$$\chi \equiv \chi_3 - \chi_2 - \chi_1.$$

The equations for $\dot{\chi}_i$ may be combined to give

$$b_1 b_2 b_3 \dot{\chi} = e^{(2\tilde{\sigma}-\sigma)t}(b_1 b_2)^2 \sin(\theta-\chi) - e^{\sigma t} b_3^2 (b_1^2 + b_2^2)\sin\chi.$$

It turns out that these equations have a very simple solution for a particular choice of initial values for b_1, b_2, b_3 and χ at $t = 0$.
Taking

$$\left.\begin{aligned} b_1(0) = b_2(0) = [2\sigma\tilde{\sigma}/\cos\chi_0\cos(\theta-\chi_0)]^{\frac{1}{2}}, \quad b_3(0) = -\tilde{\sigma}/\cos\chi_0, \\ \chi(0) = \chi_0, \quad \tan(\theta-\chi_0)\cot\chi_0 = \tilde{\sigma}/\sigma, \end{aligned}\right\} \tag{7.4}$$

it may be confirmed that the appropriate solution is

$$\chi = \chi_0, \quad b_1 = b_1(0)\,e^{-\tilde{\sigma}t}, \quad b_2 = b_2(0)\,e^{-\tilde{\sigma}t}, \quad b_3 = b_3(0)\,e^{-\sigma t}.$$

Accordingly we have

$$\left.\begin{aligned} B_1 &= [2\sigma\tilde{\sigma}/\cos\chi_0\cos(\theta-\chi_0)]^{\frac{1}{2}} \exp i[\chi_1(0) - \tilde{\sigma}t\tan\chi_0], \\ B_2 &= [2\sigma\tilde{\sigma}/\cos\chi_0\cos(\theta-\chi_0)]^{\frac{1}{2}} \exp i[\chi_2(0) - \tilde{\sigma}t\tan\chi_0], \\ B_3 &= (-\tilde{\sigma}/\cos\chi_0)\exp i[\chi_3(0) - 2\tilde{\sigma}t\tan\chi_0], \end{aligned}\right\} \tag{7.5}$$

where the initial phases satisfy

$$\chi_3(0) - \chi_2(0) - \chi_1(0) = \chi_0.$$

For physically relevant solutions $b_1(0)$, $b_2(0)$ and $b_3(0)$ must be real and positive: it is therefore necessary for both $\tilde{\sigma}^{-1}\cos\chi_0$ and $\sigma^{-1}\cos(\theta-\chi_0)$ to be negative. But from (7.4) we have $\tilde{\sigma}^{-1}\cot\chi_0 = \sigma^{-1}\cot(\theta-\chi_0)$ and it may be verified that, provided $\{(\sigma+\tilde{\sigma})^2\cot^2\theta + 4\sigma\tilde{\sigma}\} > 0$, there are four solutions for χ_0 in the range $0 \leqslant \chi_0 \leqslant 2\pi$, one of which satisfies both the above conditions. In particular, there is always such a solution when σ and $\tilde{\sigma}$ are of the same sign. The solutions (7.5) for the complex wave amplitudes B_1, B_2, B_3 are periodic in time and therefore represent purely periodic waves which undergo no temporal amplification or damping. Consequently, waves which according to linear theory are amplified or damped like $\exp(\tilde{\sigma}t)$ and $\exp(\sigma t)$ become neutrally stable for this non-linear case.

Further particular solutions may be found for the case where $\sigma = \tilde{\sigma} = 0$, corresponding to waves which are neutrally stable according to linear theory. Equations (7.3) above then simplify to

$$\left.\begin{aligned} b_1\dot{b}_1 = b_2\dot{b}_2 = b_1 b_2 b_3\cos\chi, \quad b_3\dot{b}_3 = b_1 b_2 b_3 \cos(\theta-\chi), \\ b_1 b_2 b_3\dot{\chi} = (b_1 b_2)^2\sin(\theta-\chi) - b_3^2(b_1^2+b_2^2)\sin\chi. \end{aligned}\right\} \tag{7.6}$$

When $\theta = \pi$ we have $\cos(\theta-\chi) = -\cos\chi$, $\sin(\theta-\chi) = \sin\chi$ and this set of equations becomes a special case of that examined by Simmons (1969). For this case $(b_1^2+b_2^2+2b_3^2)$ remains constant, which means that the total wave energy is conserved. The reader is referred to Simmons' paper for an exhaustive discussion of the solutions of this type.

There exists a remarkably simple particular solution of (7.6) for arbitrary θ, which is of considerable interest. This is

$$\left.\begin{aligned} \chi = \chi_0, \quad \tan\chi_0 = \tfrac{1}{2}\tan(\theta-\chi_0), \\ b_1 = b_2 = \left[\frac{\cos\chi_0}{\cos(\theta-\chi_0)}\right]^{\frac{1}{2}}\frac{b}{1-tb\cos\chi_0}, \quad b_3 = \frac{b}{1-tb\cos\chi_0}, \end{aligned}\right\} \tag{7.7}$$

where b is an arbitrary positive constant and the physically relevant roots of the equation $\tan\chi_0 = \frac{1}{2}\tan(\theta-\chi_0)$ are those for which $\cos\chi_0/\cos(\theta-\chi_0)$ is positive. Of the four roots of this equation in the range $0 \leqslant \chi_0 < 2\pi$, two have the desired property. For one of these $\cos\chi_0$ and $\cos(\theta-\chi_0)$ are both positive and for the other both are negative. For the latter root, b_1, b_2 and b_3 all decay as $(1+Kt)^{-1}$ where $K = b|\cos\chi_0|$ is a positive constant. Consequently for this solution all three waves are damped owing to a net transfer of energy from the waves to the primary shear flow. In contrast, the former root yields solutions which behave as $(1-Kt)^{-1}$ where $0 \leqslant Kt \leqslant 1$, and these attain indefinitely large amplitudes within the *finite* time $t = (b\cos\chi_0)^{-1}$. The corresponding phases χ_1, χ_2 and χ_3 are given by

$$\chi_1-\chi_1(0) = \chi_2-\chi_2(0) = \tfrac{1}{2}(\chi_3-\chi_3(0)) = -\tan\chi_0\log(1-tb\cos\chi_0),$$
$$\chi_3(0)-\chi_2(0)-\chi_1(0) = \chi_0.$$

As t approaches $(b\cos\chi_0)^{-1}$ for the root with $\cos\chi_0$ positive, these phases also change rapidly. Consequently this solution is characterized by rapidly increasing amplitudes and frequencies as t nears $b(\cos\chi_0)^{-1}$. It should be remembered that the equations (7.6) become invalid for sufficiently large amplitudes, when third and higher order terms become significant. However, this solution reveals the remarkable qualitative feature of an 'explosion' in which the wave energy grows without bound in a finite time. A similar effect, but caused by third-order terms, has recently been found by Hocking, Stewartson & Stuart (1971). One is tempted to speculate – perhaps unwisely – that such a feature may have some connexion with the final rapid development of turbulence in boundary-layer transition but it should not be forgotten that the interaction equations themselves derive from the assumption that growth rates are *small*.

8. Discussion

The foregoing analysis has established the existence of a strong non-linear mechanism for the systematic exchange of energy between the primary shear flow and the disturbance. When a resonant wave triad exists the phases of the three waves may always be such that the wave energy increases continuously by extraction of energy from the shear flow. This energy exchange has been shown to take place primarily in the vicinity of the critical layer, its effect being such that the waves propagating obliquely to the flow direction will normally grow much more rapidly than the two-dimensional wave. This is due to the surprising fact (also found in the work of Craik (1968)) that the interaction coefficient $|\Theta|$ for the former waves is of order $O(R)$, while the corresponding coefficient $|\Lambda|$ for the two-dimensional wave is $O(1)$ which is an order of magnitude smaller. This mechanism can therefore bring about the preferential amplification of three-dimensional disturbances. Also, the various particular solutions of the interaction equations examined in § 7 reveal that this growth of three-dimensionality may be remarkably rapid. Indeed, the solutions (7.7) represent a dramatic 'explosion' of wave energy, which within the limits of the approximations becomes indefinitely large in a *finite* time.

Clearly, if one pair of oblique waves grows particularly rapidly these waves will impart their characteristic spanwise periodicity to the disturbance and subsequent non-resonant interactions of the Lin–Benney type will result in correspondingly spaced secondary longitudinal-vortex flows. It is instructive to examine the experimental results of Klebanoff, Tidstrom & Sargent in the light of this hypothesis. However in doing so it should be borne in mind that the experimental observations relate to spatial instability, whereas the present theoretical model concerns only temporal growth.

It was shown in § 2 that resonant triads of the desired kind may occur among Tollmien–Schlichting waves in the unstable Blasius boundary layer. In the experiments of Klebanoff *et al.* such a boundary layer was given an artificial disturbance of prescribed frequency by means of a vibrating ribbon. This disturbance, though predominantly two-dimensional, possessed small spanwise variations in intensity, owing to the presence of small pieces of tape spaced at equal distances on the flat plate under the vibrating ribbon. As the disturbance developed downstream its three-dimensionality rapidly intensified until transition to turbulence ultimately occurred. A notable feature of the non-linear development was that, even close to the point of breakdown to turbulence, the harmonic content of the disturbance remained remarkably low. In particular the (largest) harmonic with frequency twice that of the initial disturbance contributed only about 20 % of the streamwise velocity fluctuation just before breakdown, but the disturbance was markedly three-dimensional long before this stage was reached. Accordingly, the rapidly growing three-dimensional disturbances have in the main the same frequency as the initial disturbance. Linear theory cannot convincingly account for the fact that such three-dimensional waves grow more rapidly than the two-dimensional component of the initial disturbance but the present non-linear model suggests an attractive physical explanation.

Of all possible three-dimensional waves with the given frequency there is one pair of oblique waves which will form a resonant triad with a two-dimensional wave of twice that frequency. This case is examined in § 2 for a Reynolds number of 882 and a frequency close to that of the most unstable linear disturbance. It is shown there that this pair of oblique waves have wave-number components $\alpha = 0{\cdot}23$, $\beta = \pm 0{\cdot}23$, which correspond to a propagation angle of $\frac{1}{4}\pi$. The spanwise periodicity of such waves is therefore equal to the x wavelength of the disturbance.

The experiments of Klebanoff *et al.* were mainly carried out at the greater Reynolds number (at the vibrating ribbon position) of 1635 and with an imposed disturbance of frequency 145 c/s. A two-dimensional disturbance at this frequency has $\alpha \simeq 0{\cdot}225$, $c_r \simeq 0{\cdot}34$ and a dimensional wavelength of 1·5 in. Now, an examination of the results of linear stability theory (those supplied by Dr R. Jordinson being used by the present author) reveals that at $R = 1635$ a two-dimensional Tollmien–Schlichting wave with just twice this frequency is that with $\alpha = 0{\cdot}4$, $c_r = 0{\cdot}38$. Further, the construction of a diagram similar to that of figure 2 for $R = 1635$ shows that there exists a resonant triad with wave-numbers $\alpha = 0{\cdot}4, \beta = 0$ and $\alpha = 0{\cdot}2, \beta = \pm 0{\cdot}33$, at which the oblique waves have the same frequency as the vibrating ribbon. The wave-number component $\beta = 0{\cdot}33$ is such that the spanwise spacing of the 'peaks' of a three-dimensional disturbance (comprising two such oblique waves of similar amplitude and a two-dimensional wave of the same frequency) is 1·0 in. This is precisely the artificially induced spacing in the experiments of Klebanoff *et al.* (1962, figure 2). It appears that by coincidence the initial three-dimensional disturbances in this case were just those most susceptible to resonance of the present kind.

The observation (Klebanoff *et al.* 1962, figure 6) of a tendency for the disturbance to double in frequency in the 'valleys', i.e. at the positions where the phases of the two- and three-dimensional disturbances are such that there combined amplitude is a minimum, is consistent with the proposed explanation. For, a small wave component of twice the basic frequency is required by the present model to form a resonant triad with the two oblique waves, and this component would be most easily detected at positions where the two- and three-dimensional waves with the basic frequency are in anti-phase.

However other (less detailed) observations by Klebanoff *et al.* (1962, figure 13) for an initial disturbance of frequency 65 c/s at Reynolds numbers of 1635 and 1270 at the vibrating ribbon position do not give such agreement. At this frequency a resonant triad equivalent to that just proposed yields a spanwise spacing of several inches, whereas the observed artificially-induced spacing is again 1 in.

It must be conceded, therefore, that the present model is hardly adequate for this experimental situation. But this is not surprising since the condition (6.1*b*) is violated for the disturbances introduced by the vibrating ribbon. These disturbances are much too large for the present theory to apply so it is possible that higher order effects may govern their non-linear development. On the other hand, in the early stages of natural transition the present model may be more directly relevant, but the developing disturbance may then comprise many resonant wave triads such that no definite spacing will occur. Also, in the experiments of Klebanoff *et al.* condition (6.1*a*) was not particularly well satisfied: for example, the value of $(\alpha R)^{\frac{1}{3}}$ for a 145 c/s wave at $R = 1635$ is about 7·2. At such values asymptotic linear theory still yields surprisingly good agreement with experiment. It remains to be seen whether the present non-linear theory may also provide reasonable approximations in such circumstances.

Finally, we give an *a posteriori* justification for choosing resonant wave triads of the particular form studied here. As shown above, the oblique waves of such triads have interaction coefficients which are $O(R)$. Essentially, these large coefficients derive from the fact that the critical layers of all three waves coincide at z_c. For, near z_c, the inviscid estimates for the dominant terms $(\phi_3 \hat{v}^*_{2,1} \psi_{1,2})$ of the respective integrands on the right-hand side of (3.7*b*) behave like $(z-z_c)^{-4}$, of which a factor $(z-z_c)^{-3}$ derives from $\hat{v}_2^{*\prime\prime}$ and a factor $(z-z_c)$ derives from $\psi_{1,2}$. As explained in §4, it is not then possible to deform the path of integration so as to pass either under or over the singularity at z_c since the inviscid estimates do not all remain valid on any such contour. The subsequent examination of the viscous solutions near z_c led to evaluation of the $O(R)$ contributions to the integrals. For the two-dimensional wave, on the other hand, the inviscid estimates of the integrand on the right-hand side of (3.7*a*) remain valid on a contour passing beneath the singularity at z_c and the corresponding integral is $O(1)$ in magnitude.

For any other resonant wave triads the critical layers for all these waves will not coincide. Consequently the inviscid estimates for $\hat{v}_1^{*\prime\prime}$, $\hat{v}_2^{*\prime\prime}$, ψ_3 will become singular at the different levels z_1, z_2, z_3 say. It is then possible to deform the paths of integration to pass either under or over the respective singularities at z_1, z_2, z_3 in such a way that the inviscid estimates for the integrands remain valid. As a result, these integrals will turn out to be $O(1)$ in magnitude and correspond to much weaker interactions than those for the oblique waves examined here. The present choice of wave triads has therefore focused attention upon the strongest possible resonant interactions, and such triads will comprise those waves most susceptible to rapid amplification.

In summary it may be said that the present analysis has revealed the existence of a particularly strong second-order resonance among waves in a shear flow. This has been discussed with particular reference to Tollmien–Schlichting waves in an unstable boundary layer but the basic analysis is applicable to any shear flow at large Reynolds numbers in which suitable wave triads may exist. The strength of such resonant interactions is much greater than might be anticipated owing to the remarkably large interaction coefficients of order $O(R)$. The resonant mechanism can give rise to a systematic transfer of energy from the primary shear flow to the disturbance, and this mechanism favours the growth of particular three-dimensional disturbances. Comparison of the theoretical results with existing experiments on the Blasius boundary layer is inconclusive and it is hoped that this paper may stimulate further experimented work with the aim of detecting the presence of such resonance.

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